

Quantifying the noise of a quantum channel by noise addition

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In this paper we introduce a way to quantify the noise level associated to a given quantum transformation. The key mechanism lying at the heart of the proposal is *noise addition*: in other words we compute the amount of extra noise we need to add to the system, through convex combination with a reference noisy map or by reiterative applications of the original map, before the resulting transformation becomes entanglement-breaking. We also introduce the notion of entanglement-breaking channels of order n (i.e. maps which become entanglement-breaking after n iterations), and the associated notion of amendable channels (i.e. maps which can be prevented from becoming entanglement-breaking after iterations by interposing proper quantum transformations). Explicit examples are analyzed in the context of qubit and one-mode Gaussian channels.

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I. INTRODUCTION

In quantum information several entropic functionals (the so called quantum capacities) have been introduced that provide a sort of “inverse measures” of the noise level associated with a given process, see e.g. Refs. [1–4]. In this approach the evolution of the system of interest S is described as a linear, completely positive, trace preserving mapping (CPT), the quantum channel Φ , which associates a final state $\Phi(\rho)$ to each possible initial density matrix ρ of S . The quantum capacities of Φ have a clear operational meaning as they gauge the optimal communication transmission rates achievable when operating in parallel on multiple copies of S : consequently the noisier the channel is, the lower are its associated quantum capacities. Unfortunately however, even for small systems, these quantities are also extremely difficult to evaluate since require optimization over large coding spaces, e.g. see Ref. [5, 6].

In this paper we introduce an alternative way to determine how disruptive a channel might be which, while still having a simple operational interpretation, it is easier to compute than the quantum capacities. The starting point of our analysis is to use Entanglement-Breaking (EB) channels [7, 8] as the fundamental benchmarks for evaluating the noise level of a transformation. We remind that a map operating on a system S , is said to be entanglement-breaking if for all its extensions to an ancillary system A it annihilates the entanglement between the system and the ancilla [7, 8]. From the point of view of quantum information, the action of these maps represents hence the most disruptive form of noise a quantum mechanical system can undergo. A reasonable way to quantify the noise level of a generic map Φ can then be introduced by computing how much extra noise we need to “add” to it before the resulting transformation becomes entanglement-breaking. The intuitive idea behind this approach is that channels which are less disruptive on S should require larger amount of extra noise to behave like an entanglement-breaking map.

In the following we analyze two different mechanisms

of noise addition. The first one assumes to form convex combinations $(1 - \mu)\Phi + \mu\Phi_{\text{DEP}}^{\rho_0}$ of the input channel Φ with generalized completely depolarizing channels $\Phi_{\text{DEP}}^{\rho_0}$ (these are the most drastic examples of entanglement-breaking channels which bring every state of S into a unique output configuration ρ_0 – the fixed point of the map). In this approach the level of noise associated with the original map Φ is gauged by the minimum value μ_c of the mixing parameter μ which transforms $(1 - \mu)\Phi + \mu\Phi_{\text{DEP}}^{\rho_0}$ into an entanglement-breaking map (as we will see a proper characterization of this measure requires an optimization upon ρ_0).

The second mechanisms of noise addition we consider assumes instead the reiterative application of Φ on S . In this case the noise level is determined by the minimum value n_c of iterations needed to transform Φ into an entanglement-breaking map (if such minimum exists). As we shall see, due to the lack of monotonicity under concatenation with other maps, this second functional cannot be considered a proper measure of the noise level introduced by Φ (regularized version of n_c *do* however retain this property). Nonetheless n_c captures some important aspects of the dynamics associated with Φ : namely it counts the number of discrete time evolutions induced by the map that a system can sustain before its entanglement with an external ancilla is completely destroyed. The definition of n_c gives us also the opportunity of introducing the set EB^n of the entanglement-breaking channels of order n , and the notion of *amendable* channels. The former is composed by all CPT maps Φ which, when applied n times, are entanglement-breaking. Vice-versa a channel Φ is amendable if it can be prevented from becoming entanglement-breaking after n_c iterations via a proper application of intermediate quantum channels – a similar problem was also discussed in Ref. [9].

In the following we will discuss some general features of the functional μ_c and n_c computing their exact values for some special class of channels. Specifically in Sec. II we introduce the functionals and characterize some general properties. In Sec. III we will restrict our attention to qubit channels computing the value of μ_c and n_c for the

set of unital maps and for the generalized amplitude-damping channels. Also using examples from these sets we will show that the set EB^n is not convex for $n \geq 2$ and provide evidence of the existence of amendable maps. In Sec. IV we will consider the analogous of qubit maps for continuous variable systems, that is one-mode gaussian channels, and in particular we will evaluate the functional n_c for attenuation and amplification channels. Conclusions and final remarks are given in Sec. V.

II. DEFINITIONS AND BASIC PROPERTIES

It is a well known [7, 8] fact that any entanglement-breaking channel Φ_{EB} can be described as measure and re-prepare scheme according to which the system S is initialized in some state ρ_ℓ depending on the outcome ℓ of a measurement performed on the input state ρ , i.e.

$$\Phi_{\text{EB}}[\rho] = \sum_{\ell} \rho_{\ell} \text{Tr}[F_{\ell} \rho] , \quad (1)$$

(here $\{F_{\ell}\}$ are the elements of the positive operator valued measure (POVM) which defines the measuring process). These maps cannot be used to convey quantum information even in the presence of two-way classical side channel (of course they might allow to achieve this goal when the communicating parties are given some prior shared entanglement via teleportation). In this respect they can thus be considered as “classical” communication lines – notice however that they might be used to create *quantum discord* [10]. One can easily verify that the set EB formed by the entanglement-breaking channels is closed under convex combinations and under sequential concatenation, i.e. given $p \in [0, 1]$ and $\Phi'_{\text{EB}}, \Phi''_{\text{EB}} \in \text{EB}$ we have $p\Phi'_{\text{EB}} + (1-p)\Phi''_{\text{EB}} \in \text{EB}$ and $\Phi'_{\text{EB}} \circ \Phi''_{\text{EB}} \in \text{EB}$ [here “ \circ ” stands for the composition of super-operators defined in Sec. II A 1]. More generally, it is also true that the set EB is stable under concatenation with other (not necessarily entanglement-breaking) maps, i.e.

$$\Phi_{\text{EB}} \in \text{EB} \implies \Phi \circ \Phi_{\text{EB}}, \Phi_{\text{EB}} \circ \Phi \in \text{EB} , \quad \forall \Phi \text{ CPT.} \quad (2)$$

A. Noise addition via convex convolution

A proper subset of the EB channels is provided by the completely depolarizing maps, which transform any input state ρ of S into an assigned fixed point $\rho_0 \in \mathfrak{S}(\mathcal{H}_S)$, i.e.

$$\Phi_{\text{DEP}}^{\rho_0}[\rho] = \rho_0 \text{Tr}[\rho] . \quad (3)$$

(we indicate with $\mathfrak{S}(\mathcal{H}_S)$ the set of density matrices of S). In a sense, these maps form the hard core of the EB set which prevents any sort of communication (not even classical). We can thus use them as a fundamental “unity” of added noise. Let then Φ be a generic map acting on S . For each completely depolarizing channel $\Phi_{\text{DEP}}^{\rho_0}$ we define $\mu(\Phi; \rho_0)$ to be the minimum value of the mixing

probability parameter $\mu \in [0, 1]$ that transforms the convex convolution $(1 - \mu)\Phi + \mu\Phi_{\text{DEP}}^{\rho_0}$ into an element of EB , i.e.

$$\mu(\Phi; \rho_0) := \min_{\mu \in [0, 1]} \{ (1 - \mu)\Phi + \mu\Phi_{\text{DEP}}^{\rho_0} \in \text{EB} \} . \quad (4)$$

Thanks to the Choi-Jamiolkowski isomorphism [11], computing $\mu(\Phi; \rho_0)$ corresponds to determining the minimum μ for which the state

$$\Gamma_{\rho_0, \mu}^{\Phi} = (1 - \mu)(\Phi \otimes I)[\psi_+] + \mu \rho_0 \otimes \frac{\mathbb{1}}{d} \quad (5)$$

is separable. Here we used the symbol ψ_+ to represent the density matrix $|\psi_+\rangle\langle\psi_+|$ associated with the maximally entangled state $|\psi_+\rangle = \frac{1}{\sqrt{d}} \sum_{j=1}^d |j\rangle \otimes |j\rangle \in \mathcal{H}_S^{\otimes 2}$, $\{|j\rangle\}$ being an orthonormal basis of the Hilbert space of the system S , and $d = \dim \mathcal{H}_S$ being its dimension. We remind that the specific choice of such basis is completely irrelevant when constructing the Choi-Jamiolkowski state.

The quantity (4) measures the amount of $\Phi_{\text{DEP}}^{\rho_0}$ we need to mix to Φ via a classical stochastic process in order to make the resulting map entanglement-breaking. Clearly $\mu(\Phi; \rho_0)$ nullifies if Φ is already an element of EB . Vice-versa for non entanglement-breaking channels Φ the value of $\mu(\Phi; \rho_0)$ is always non null and in general might depend on the selected $\Phi_{\text{DEP}}^{\rho_0}$ (some of the properties of $\mu(\Phi; \rho_0)$ are discussed in Appendix A). To get a functional of Φ alone we need hence to optimize with respect to the possible choices of the fix point ρ_0 . This brings us to define the function

$$\mu_c(\Phi) := \min_{\rho_0} \mu(\Phi; \rho_0) , \quad (6)$$

which constitutes our first (inverse) measure of the noise level associated with Φ . Solving the minimization in Eq. (6) is not simple in general: still in the next section we shall see that for some classes of channels which possess special symmetries this is feasible. Here we present some basic properties of the functional $\mu_c(\Phi)$.

1. Monotonicity under Concatenation

Given Φ and Ψ CPT transformations we have

$$\mu_c(\Phi \circ \Psi) \leq \mu_c(\Psi) , \quad (7)$$

$$\mu_c(\Phi \circ \Psi) \leq \mu_c(\Phi) , \quad (8)$$

with “ \circ ” being the composition of super-operators such that for all ρ we have $(\Phi \circ \Psi)[\rho] = \Phi[\Psi[\rho]]$. Property (7) is almost straightforward. Indeed for all $\mu \geq \mu_c(\Psi)$ there exists a density matrix $\rho_{\Psi, \mu}$ such that $\Gamma_{\rho_{\Psi, \mu}}^{\Psi}$ is separable. Since for every $\Phi \in \text{CPT}$ the state $(\Phi \otimes I)\Gamma_{\rho_{\Psi, \mu}}^{\Psi}$ is still separable, $\mu \geq \mu_c(\Phi \circ \Psi)$ which implies $\mu_c(\Phi \circ \Psi) \leq \mu_c(\Psi)$. Property (8) can be proved analogously. Indeed

if $\mu \geq \mu_c(\Phi)$ there exists a depolarizing channel $\Phi_{\text{DEP}}^{\rho_\Phi}$ such that

$$(1 - \mu)(\Phi \otimes I) + \mu(\Phi_{\text{DEP}}^{\rho_\Phi} \otimes I) \in \text{EB}. \quad (9)$$

Thus, if we apply this map to the state $(\Psi \otimes I)(\psi_+)$ we get a separable state, and property (8) is immediately proved.

An almost trivial consequence of the above inequalities is the fact that μ_c is constant under unitary transformations, i.e.

$$\mu_c(\mathcal{U} \circ \Phi \circ \mathcal{V}) = \mu_c(\Phi), \quad (10)$$

where \mathcal{U} and \mathcal{V} stands for unitary channels, i.e. $\mathcal{U}(\rho) := U\rho U^\dagger$ with U being unitary.

2. Extremal values

By definition $\mu_c(\Phi)$ is always positive (it nullifies if and only if $\Phi \in \text{EB}$) and smaller than 1. As a matter of fact, it is possible to show that

$$\mu_c(\Phi) \leq \frac{d}{1+d}, \quad (11)$$

the upper bound being achieved for noiseless (unitary) channels. To see this we first use the fact that the generalized Werner states

$$\Gamma_{\mathbb{1}/d,\mu}^I := (1 - \mu)(I \otimes I)[\psi_+] + \mu \frac{\mathbb{1}}{d} \otimes \frac{\mathbb{1}}{d} \quad (12)$$

are separable iff $\mu \geq d/(1+d)$ [12]. Therefore since $\Gamma_{\mathbb{1}/d,\mu}^I$ is the Choi-Jamiolkowski state of the identity channel $\Phi = I$ this implies,

$$\mu(I; \mathbb{1}/d) = d/(1+d). \quad (13)$$

Now the bound (11) can be easily established by noticing that

$$\mu_c(\Phi) \leq \mu(\Phi; \Phi(\mathbb{1})/d) \leq \mu(I; \mathbb{1}/d) = d/(1+d), \quad (14)$$

where the first inequality is just a consequence of the definition (6) while the second one follows from the fact that the Choi-Jamiolkowski state $\Gamma_{\Phi(\mathbb{1})/d,\mu}^\Phi$ can be obtained from $\Gamma_{\mathbb{1}/d,\mu}^I$ via the application of a local channel, i.e. $\Gamma_{\Phi(\mathbb{1})/d,\mu}^\Phi = (\Phi \otimes I)[\Gamma_{\mathbb{1}/d,\mu}^I]$. Remains to show that the threshold value (6) is achievable for unitary channels, i.e.

$$\mu_c(\mathcal{U}) = d/(1+d). \quad (15)$$

Thanks to the property (10) we can just focus on the case $\mathcal{U} = I$ for which we have already established the condition (13). To prove the relation (15) we need hence only to verify that $\mu(I; \mathbb{1}/d) \leq \mu(I; \rho_0)$ for all ρ_0 . This can be proven by observing that $\Gamma_{\mathbb{1}/d,\mu}^I$ is obtained from

$\Gamma_{\rho_0,\mu}^I$ via the application of the twirling isotropic map [13, 14]

$$\mathcal{P}[\dots] := \int dm(V)(V \otimes V^*)[\dots](V \otimes V^*)^\dagger, \quad (16)$$

where the integral is evaluated over the elements V of the unitary group on \mathcal{H}_S , $dm(V)$ is the Haar measure, while V^* represents the unitary transformation obtained from V by complex conjugation with respect to the canonical basis $\{|j\rangle\}$. Indeed, on one hand $|\psi_+\rangle$ is invariant under the application of $V \otimes V^*$ (i.e. $V \otimes V^*|\psi_+\rangle = |\psi_+\rangle$) while on the other hand $\int dm(V)V\rho_0V^\dagger = \mathbb{1}/d$, so that

$$\mathcal{P}(\Gamma_{\rho_0,\mu}^I) = \Gamma_{\mathbb{1}/d,\mu}^I. \quad (17)$$

Take then $\mu = \mu(I; \rho_0)$. With this choice the state $\Gamma_{\rho_0,\mu}^I$ is separable. Since \mathcal{P} is LOCC [13], also $\Gamma_{\mathbb{1}/d,\mu}^I$ is separable. Therefore we must have $\mu(I; \mathbb{1}/d) \leq \mu$ concluding the derivation.

3. Convexity rules

Consider $\{p_j; \Phi_j\}$ a statistical ensemble of CPT channels Φ_j distributed according to the probabilities p_j . Then given $\mu_j := \mu_c(\Phi_j)$ the value assumed by the functional μ_c on the j -th element of the ensemble, we have that

$$\min_j \mu_j \leq \mu_c \left(\sum_j p_j \Phi_j \right) \leq \bar{\mu}, \quad (18)$$

where

$$\bar{\mu} = \frac{\sum_j (\mu_j p_j)/(1 - \mu_j)}{\sum_k p_k/(1 - \mu_k)} \leq \max_j \mu_j. \quad (19)$$

The proof of the first inequality trivially follows from the fact that convex combinations of separable states are still separable. To verify the second one instead it is convenient to introduce the probabilities $q_j := (\frac{p_j}{1 - \mu_j})/(\sum_k \frac{p_k}{1 - \mu_k})$ and the states ρ_j associated to the optimal depolarizing channel which saturates the minimization (6) for the map Φ_j . Then the result simply follows by noticing that the density matrix

$$\sum_j q_j \Gamma_{\rho_j,\mu_j}^{\Phi_j} = (1 - \bar{\mu})(\Phi \otimes I)[\psi_+] + \bar{\mu}(\rho_0 \otimes \mathbb{1}/d) \quad (20)$$

is separable (here ρ_0 is the state $\sum_j \frac{q_j \mu_j}{\bar{\mu}} \rho_j$).

Equations (18) and (19) state that the value of μ_c associated with the average map $\sum_j p_j \Phi_j$ of $\{p_j; \Phi_j\}$ always lays between the extremal values it takes on the elements of the ensemble.

B. Noise addition via concatenation

As evident from Sec. II A 1 a simple method for increasing the noise level of a given map Φ is via concatenation. In particular Eq. (7) implies that

$$\mu_c(\Phi^n) \leq \mu_c(\Phi^{n-1}) \leq \dots \leq \mu_c(\Phi^2) \leq \mu_c(\Phi), \quad (21)$$

where for each n integer we have set

$$\Phi^n := \underbrace{\Phi \circ \Phi \circ \dots \circ \Phi}_{n \text{ times}}. \quad (22)$$

This allows us to introduce a new criterion for classifying noisy quantum channels. For this purpose, given n integer we define EB^n to be the set formed by the CPT maps Φ acting on S such that Φ^n is entanglement-breaking, i.e. explicitly

$$\text{EB}^n := \{\Phi \in \text{CPT}, \text{ s. t. } \mu_c(\Phi^n) = 0\}. \quad (23)$$

[Note: The set EB^2 was introduced in [15] in connection to the set of 2-local entanglement-annihilating channels]. From Eq. (21) it follows that for all $m \geq n$ we have

$$\text{EB}^n \subset \text{EB}^m, \quad (24)$$

the inclusion being strict (explicit examples of maps belonging to EB^m but not to EB^n will be provided in the next sections). Clearly a channel Φ which is an element of EB^{n+1} but which does not belong to EB^n can be considered “less noisy” than the elements of the latter set as they break the entanglement shared between the system and an arbitrary ancilla just after n reiterations. This motivates the introduction of a new functional $n_c(\Phi)$ to “measure” the noise level of Φ . In particular if $\Phi \in \text{EB}$ we set $n_c(\Phi) = 1$. Vice-versa, if for an integer number $n \geq 1$ we have $\Phi \in \text{EB}^n/\text{EB}^{n-1}$, we say that the map Φ is an entanglement-breaking channel of order n , and we set $n_c(\Phi) = n$. Finally if for all n integers, Φ does not belong to $\text{EB}^n/\text{EB}^{n-1}$, then we set $n_c(\Phi) = \infty$. Trivial examples of channels with $n_c(\Phi) = \infty$ are the unitary transformations. Other maps which do not become entanglement-breaking after any number of iterations are the amplitude-damping channels – see Sec. III B 2 for details.

1. Amendable channels

As anticipated in Sec. I, differently from μ_c of Sec. II A, the functional n_c is not monotonic under concatenation, i.e. it is *not* true that for an arbitrary CPT channels Ψ one has $n_c(\Psi \circ \Phi) \leq n_c(\Phi)$. Notice however that from Eq. (24) it trivially follows that for each $\Phi \in \text{CPT}$ and n integer we still have

$$n_c(\Phi^n) \geq n_c(\Phi^{n+1}). \quad (25)$$

Furthermore exploiting the fact that the entanglement-breaking property of a map is invariant under unitary

redefinition of the input and output spaces, and the fact that $(\mathcal{U} \circ \Phi \circ \mathcal{U}^\dagger)^n = \mathcal{U} \circ \Phi^n \circ \mathcal{U}^\dagger$ we also have

$$n_c(\mathcal{U} \circ \Phi \circ \mathcal{U}^\dagger) = n_c(\Phi), \quad (26)$$

\mathcal{U}^\dagger being the inverse of \mathcal{U} .

The absence of monotonicity under concatenation implies that n_c will be typically *not* invariant under unitary equivalence built upon uncorrelated unitary redefinitions of the input and output subspaces, i.e. exists Φ and \mathcal{U} , \mathcal{V} unitaries such that

$$n_c(\mathcal{U} \circ \Phi \circ \mathcal{V}) \neq n_c(\Phi), \quad (27)$$

(compare this with the identity (26)). These facts are strictly related with the notion of *amendable channels*. Specifically we say that a map Φ is amendable if there exists a *filtering* CPT map $\tilde{\Phi}$ such that $\Phi \circ \tilde{\Phi} \circ \Phi \notin \text{EB}$ being $\Phi^2 \in \text{EB}$ [16]. In particular we will focus on unitary filters $\tilde{\Phi}$. Indeed explicit examples can be found (see next section) of transformations $\Phi \in \text{EB}^2/\text{EB}$ for which there exists \mathcal{V} unitary such that $\Phi \circ \mathcal{V} \circ \Phi$ (and therefore its unitarily conjugate $\mathcal{V} \circ \Phi \circ \mathcal{V} \circ \Phi$) is not entanglement-breaking. Accordingly, the following inequality holds,

$$2 = n_c(\Phi) < n_c(\mathcal{V} \circ \Phi), \quad (28)$$

which explicitly disproves the monotonicity of n_c under concatenation.

The notion of amendable channels will be clarified in Sec. III where, focusing on unital and generalized amplitude-damping channels for qubits, we will provide explicit examples of the inequality (28). In particular in Sec. III B 2 we will show that, given an arbitrary integer $m \geq 2$, it is possible to find Φ and \mathcal{V} such that (28) holds with $n_c(\mathcal{V} \circ \Phi) = m$ (maps Φ for which this is possible will be called *amendable channel of order m*). Such examples show that there are cases where even though two successive uses of the same channel Φ are sufficient to destroy completely the entanglement present in the system, one can *delay* such detrimental effect by $m - 2$ steps by simply acting with the same intermediate filtering rotation \mathcal{V} after each channel application. In other words, for these special maps it is possible to make the sequences

$$\underbrace{(\mathcal{V} \circ \Phi) \circ (\mathcal{V} \circ \Phi) \circ \dots \circ (\mathcal{V} \circ \Phi)}_{m' \text{ times}}, \quad (29)$$

not entanglement-breaking for all $m' < m$.

2. Regularization of n_c

Property (27) implies that the functional n_c cannot be considered a proper *measure of noise* for quantum maps but rather a *criterion* for classifying them. More stable versions of n_c can be obtained by optimizing with respect to all filtering transformations. For instance a possibility is to consider the regularization

$$\bar{n}_c(\Phi) := \max_{\Psi} n_c(\Psi \circ \Phi), \quad (30)$$

where the maximization is performed over all CPT maps Ψ . By construction (30) is explicitly monotonic under concatenation, i.e.

$$\bar{n}_c(\Phi_2 \circ \Phi_1) \leq \bar{n}_c(\Phi_1), \quad (31)$$

for all Φ_1, Φ_2 CPT. Notice however that it is still possible that *not-uniform* filtering CPT transformations $\mathcal{V}_1, \mathcal{V}_2, \dots, \mathcal{V}_m$ might exist which make the map

$$(\mathcal{V}_m \circ \Phi) \circ (\mathcal{V}_{m-1} \circ \Phi) \circ \dots \circ (\mathcal{V}_1 \circ \Phi), \quad (32)$$

not EB for $m \geq \bar{n}_c(\Phi)$. Removing this last instability is possible (e.g. adopting the *quantum comb* formalism of Ref. [17]) but clearly the resulting expression becomes extremely complex. Furthermore any optimization with respect to the filtering transformations tends to hide some of the structural properties of Φ . For this reason, in what follows we will restrict our attention to compute the non-regularized version of n_c introduced in Sec. II B.

III. QUBIT CHANNELS

In what follows we analyze the behavior of the noise-quantifiers introduced in the previous sections by focusing on two important classes of qubit channels which, due to their special symmetries, allow us to provide closed expressions for both μ_c and n_c . We start in Sec. III A by studying the set of qubit unital maps [18]. Here we will construct explicit examples of amendable channels of order 2 and prove that the sets EB^n are not convex for $n \geq 2$. In Sec. III B instead we will focus on generalized amplitude-damping channels which will allow us to show that exist examples of amendable channels of arbitrary order.

A. Unital channels

We recall that unital CPT channels transform the identity operator into itself, i.e. $\Phi[\mathbb{1}] = \mathbb{1}$. This set includes unitary transformations as a proper subset, and it is closed under closed convex combination and channel multiplication [1]. When operating on a single qubit, unital channels admit a simple one-to-one parameterization

$$\Phi \leftrightarrow T, \quad (33)$$

in terms of the 3×3 real matrices T which fulfills the necessary and sufficient constraint [18]

$$T^\dagger T \leq \mathbb{1}, \quad (34)$$

(the inequality being saturated by the T which correspond to unitary transformations – in other words unitary qubit channels can be uniquely associated with 3×3 real matrices O which are orthogonal, i.e. $O^\dagger O = \mathbb{1}$).

The correspondence (33) can be easily established by expanding the system density operators in the canonical operator basis $\{\mathbb{1}, \sigma_1, \sigma_2, \sigma_3\}$ given by the unit $\mathbb{1}$ and the Pauli matrices σ_i , $1 \leq i \leq 3$. Accordingly, given a generic qubit density matrix $\rho = (\mathbb{1} + \vec{v} \cdot \vec{\sigma})/2 \in \mathfrak{S}(\mathcal{H}_S)$ with $\vec{v} \in \mathbb{R}^3$ such that $|\vec{v}| \leq 1$, we can express its evolved counterpart $\Phi(\rho) = (\mathbb{1} + \vec{v}' \cdot \vec{\sigma})/2$ via the unital CPT channel Φ as the linear mapping

$$\vec{v} \rightarrow \vec{v}' = T\vec{v}, \quad (35)$$

with T being 3×3 real matrix fulfilling (34). It also follows that given two unital maps Φ_1 and Φ_2 , characterized by the 3×3 real matrices T_1 and T_2 respectively, the composite map $\Phi_1 \circ \Phi_2$ is associated to the matrix $T = T_1 T_2$, i.e.

$$\Phi_1 \circ \Phi_2 \leftrightarrow T_1 T_2. \quad (36)$$

Within the correspondence (33) it was shown [18] that the entanglement-breaking property of a unital qubit channel Φ only depends on the singular values of T . More precisely, a unital map is EB if and only if

$$\|T\|_1 = \text{Tr}[\Lambda] \leq 1, \quad \Lambda := |T| = \sqrt{T^\dagger T}, \quad (37)$$

where $\|\cdot\|_1$ is the trace norm and corresponds to the sum of the singular values of the matrix it is applied to, while the real non-negative matrix Λ is related to T via polar decomposition,

$$T = O\Lambda, \quad (38)$$

with O being a 3×3 orthogonal matrix [19]. Observe that Λ satisfies the condition (34) and can be hence associated with a unital qubit CPT channel via the correspondence (33). As a matter of fact this new map Φ_p is the *polar form* of the original channel Φ which is related to the latter through a unitary transformation according to the identity

$$\Phi = \mathcal{U} \circ \Phi_p, \quad (39)$$

which mimics (38) at the super-operator level via the identity (36) (here \mathcal{U} is the unitary channel which is associated to the orthogonal matrix O of Eq. (38) through the correspondence (33)). The above construction in particular shows also that, without loss of generality, when characterizing the entanglement-breaking property of a map Φ we can just focus on its polar form Φ_p . At the level of superoperators this follows simply from (39) and by the fact that channels which differ by unitary transformations share the same entanglement-breaking property (alternatively this can also be verified through Eq. (37) and noticing that the singular values of Λ coincides with those of T so that $\|\Lambda\|_1 = \|T\|_1$).

The decomposition (38) can be further simplified by expressing T as

$$T = O_1 D O_2, \quad (40)$$

where $O_{1,2}$ are orthogonal matrices, while $D = \text{diag}\{\lambda_1, \lambda_2, \lambda_3\}$ is a diagonal (not necessarily positive) matrix with elements λ_j , which in modulus correspond to the singular values of T (i.e. to the eigenvalues of Λ) so that

$$\|T\|_1 = \sum_{i=1}^3 |\lambda_i|. \quad (41)$$

In terms of the equivalence (33), the matrix D defines a unital channel Φ_λ which gives the *canonical representation* of the map Φ [18], and which is related to the latter through the identity

$$\Phi = \mathcal{U}_1 \circ \Phi_\lambda \circ \mathcal{U}_2, \quad (42)$$

with $\mathcal{U}_{1,2}$ being unitary transformations associated to the matrix $O_{1,2}$ through the isomorphism (33). As in the case of the polar representation, it is clear that Φ (or Φ_p) is entanglement-breaking if and only if Φ_λ is entanglement-breaking, i.e.

$$\Phi \in \text{EB} \Leftrightarrow \Phi_p \in \text{EB} \Leftrightarrow \Phi_\lambda \in \text{EB}. \quad (43)$$

1. Noise addition via convex convolution

For a unital qubit channel Φ described by the matrix T , the value $\mu_c(\Phi)$ can be exactly computed resulting in the following increasing function of $\|T\|_1$,

$$\mu_c(\Phi) = \max \left\{ 0, \frac{\|T\|_1 - 1}{\|T\|_1} \right\}, \quad (44)$$

which for unitary transformations gives $\mu_c(\Phi) = 2/3$ in agreement with Eq. (15) (in this case in fact $\|T\|_1 = 3$), while for entanglement-breaking channels it correctly gives $\mu_c(\Phi) = 0$ (in this case in fact $\|T\|_1 \leq 1$).

To derive Eq. (44) we first exploit the property (10) and the identity (42) to write

$$\mu_c(\Phi) = \mu_c(\Phi_p) = \mu_c(\Phi_\lambda). \quad (45)$$

Now the value of $\mu_c(\Phi_\lambda)$ can be directly computed by noticing that

$$\mu_c(\Phi_\lambda) = \mu(\Phi_\lambda; \mathbb{1}/2). \quad (46)$$

Proving this identity requires us to show that if there exists $\rho_0 \in \mathfrak{S}(\mathcal{H}_S)$ such that for a given $\mu \in [0, 1]$, the Choi-Jamiolkowski state (5) $\Gamma_{\rho_0, \mu}^{\Phi_\lambda}$ is separable, then for that same μ , the state $\Gamma_{\mathbb{1}/2, \mu}^{\Phi_\lambda}$ is also separable. This follows by the fact that the canonical channels Φ_λ satisfy the following relation (see Appendix B),

$$\Phi_\lambda = \mathcal{S}_j^\dagger \circ \Phi_\lambda \circ \mathcal{S}_j, \quad (47)$$

where for $j \in \{0, 1, 2, 3\}$ the symbol $\mathcal{S}_j = \mathcal{S}_j^\dagger$ stands for the unitary mapping

$$\mathcal{S}_j[\rho] := \sigma_j \rho \sigma_j, \quad (48)$$

(here we used $\sigma_0 := \mathbb{1}$). Suppose then that for some ρ_0 and μ the state $\Gamma_{\rho_0, \mu}^{\Phi_\lambda}$ is separable. This implies that also $(\mathcal{S}_j \otimes \mathcal{S}_j)(\Gamma_{\rho_0, \mu}^{\Phi_\lambda})$ is separable for all j (indeed this is just $\Gamma_{\rho_0, \mu}^{\Phi_\lambda}$ transformed under local rotations). Notice that such state can be expressed as

$$\begin{aligned} (\mathcal{S}_j \otimes \mathcal{S}_j)(\Gamma_{\rho_0, \mu}^{\Phi_\lambda}) &= (1 - \mu)(\Phi_\lambda \circ \mathcal{S}_j \otimes \mathcal{S}_j)[\psi_+] \\ &\quad + \mu(\mathcal{S}_j[\rho_0] \otimes \frac{\mathbb{1}}{2}) \\ &= (1 - \mu)(\Phi_\lambda \otimes I)[\psi_+] + \mu(\mathcal{S}_j[\rho_0] \otimes \frac{\mathbb{1}}{2}), \end{aligned} \quad (49)$$

where in the first line we used Eq. (47) and the fact that $\mathcal{S}_j \circ \mathcal{S}_j = I$, while in the last one we used the fact that $(\mathcal{S}_j \otimes \mathcal{S}_j)[\psi_+] = \psi_+$ [20]. Next we observe that $\Gamma_{\mathbb{1}/2, \mu}^{\Phi_\lambda}$ can be obtained from (49) by averaging over j (indeed $\sum_{j=0}^3 \mathcal{S}_j/4 = \Phi_{\text{DEP}}^{\mathbb{1}/2}$ and hence $\sum_{j=0}^3 \mathcal{S}_j[\rho_0]/4 = \mathbb{1}/2$). Therefore, since mixtures of separable states are separable, we conclude that $\Gamma_{\mathbb{1}/2, \mu}^{\Phi_\lambda}$ is separable, proving Eq. (46). The final step in deriving Eq. (44) is hence the evaluation of $\mu(\Phi_\lambda; \mathbb{1}/2)$. This however is easily obtained by noticing that $(1 - \mu)\Phi_\lambda + \mu\Phi_{\text{DEP}}^{\mathbb{1}/2}$ is a unital channel which at the level of the correspondence (33) is characterized by a matrix T' having singular values $|\lambda'_j| = (1 - \mu)|\lambda_j|$. Therefore according to Eq. (37) it is entanglement-breaking for

$$\sum_{i=1}^3 (1 - \mu)|\lambda_i| \leq 1 \Leftrightarrow \mu \geq \frac{\sum_{i=1}^3 |\lambda_i| - 1}{\sum_{j=1}^3 |\lambda_j|} = \frac{\|T\|_1 - 1}{\|T\|_1}, \quad (50)$$

yielding

$$\mu(\Phi_\lambda; \mathbb{1}/2) = \max \left\{ 0, \frac{\|T\|_1 - 1}{\|T\|_1} \right\}, \quad (51)$$

and hence concluding the calculation.

2. Noise addition via concatenation and amendable unital channels

For what concerns the mechanism of noise addition via concatenation, we start noting that differently from the previous section, a unital qubit channel Φ and its polar Φ_p or canonical Φ_λ forms might not share the same properties: indeed, as anticipated in Sec. II B 1, the functional n_c is not invariant under unitary equivalence (an explicit counterexample will be provided in the following). Still it turns out that Φ_p and Φ_λ have same the value of n_c [21] and that a sufficient (but not necessary) condition for a generic unital channel Φ to be EB^n is $\Phi_p \in \text{EB}^n$, implying

$$n_c(\Phi_p) \leq n_c(\Phi) = n_c(\mathcal{U} \circ \Phi_p). \quad (52)$$

To prove this relation we first notice that from Eqs. (36) and (37) it follows that Φ^n is entanglement-breaking if

and only if

$$\|T^n\|_1 \leq 1, \quad (53)$$

where T is the matrix associated to Φ via the correspondence (33). Hence, reminding Eq. (25) we can write

$$n_c(\Phi) = \min_n \{n \text{ s.t. } \|T^n\|_1 \leq 1\}, \quad (54)$$

where the minimization is performed over all n integers (notice that for entanglement-breaking channels this yields correctly $n_c(\Phi) = 1$). Equation (52) can then be derived by noticing that the matrices T and Λ which represent Φ and Φ_p , respectively, are related as in (38) and by using the Hölder inequality [22]. Indeed

$$\begin{aligned} \|T^n\|_1 = \|(O\Lambda)^n\|_1 &\leq (\|O\Lambda\|_n)^n = (\|\Lambda\|_n)^n \\ &= \left(\text{Tr}[\Lambda^n]^{1/n}\right)^n = \|\Lambda^n\|_1, \end{aligned} \quad (55)$$

where $\|\cdot\|_n$ is the operator n -norm, and where we used the fact that O is orthogonal and Λ non-negative. From this we can hence conclude that if $n_c(\Phi_p) = n$ (that is $\|\Lambda^n\|_1 \leq 1$), then $\Phi = \mathcal{U} \circ \Phi_p \in \text{EB}^n$ too ($\|(O\Lambda)^n\|_1 \leq 1$), concluding the proof.

On the contrary, for $n \geq 2$ there exist $\Phi \in \text{EB}^n$ such that their polar form Φ_p is not in EB^n [23]. As an example consider for instance the case in which Φ_p is characterized by the matrix

$$\Lambda = \begin{bmatrix} 0.73 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad (56)$$

which satisfies the condition (34) and for which one has

$$\begin{aligned} \|\Lambda\|_1 &= 1.73 > 1, & \|\Lambda^2\|_1 &\simeq 1.03 > 1, \\ \|\Lambda^3\|_1 &\simeq 0.64 < 1, \end{aligned} \quad (57)$$

that imply $n_c(\Phi_p) = 3$ and hence $\Phi_p \in \text{EB}^3/\text{EB}^2 \subset \text{EB}^3$. Now construct Φ as the unital map associated with the matrix T defined as in Eq. (38) where O is the orthogonal matrix

$$O = O^\dagger = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (58)$$

that is

$$T = O\Lambda = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.73 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}. \quad (59)$$

With this choice we have

$$\|T^2\|_1 = 0.98 < 1, \quad (60)$$

which gives $n_c(\Phi) = 2$ and thus $\Phi \in \text{EB}^2/\text{EB} \subset \text{EB}^2$.

The same example proves the existence of unital qubit maps that are amendable. To see this explicitly take

\mathcal{V} of Eq. (28) as the inverse \mathcal{U}^\dagger of the unitary \mathcal{U} of Eq. (39) and notice that while the map Φ introduced above is an element of EB^2 (i.e. $\Phi^2 \in \text{EB}$), the channel $\Phi \circ \mathcal{U}^\dagger \circ \Phi = \mathcal{U} \circ \Phi_p^2$ is not entanglement-breaking (it is indeed unitarily equivalent to Φ_p^2 which is not in EB according to Eq. (57)): this implies that, by interposing a unitary transformation, we can prevent two consecutive applications of Φ of becoming entanglement-breaking. For the sake of completeness is worth mentioning that this fact is in contradiction with one of the claims of Ref. [9] where it was stated (without presenting a formal proof) that unital qubit channels are not rectifiable (and hence not amendable). At the origin of this incompatibility between such claim and our findings, we believe is the fact that the Authors of [9] assumed the possibility of always putting the channel in the canonical representation (42), without considering the fact that the properties of noise addition via concatenation are *not* invariant under unitary equivalence.

3. Non convexity of the sets EB^n

The set of EB channels is known to be convex. The same property however does not apply for the set EB^n with $n \geq 2$. In what follows we will show this fact by presenting counter-examples taken from the unital qubit maps for $n = 2, 3$ (even though we did not check for larger n we believe that the similar counter-examples can be found also for those cases). Specifically we will prove that even though convex combinations of unital channels are still unital, the unital qubit channels which are in EB^2 and in EB^3 are not closed under convex convolution.

a. Non convexity of EB^2 : Via the correspondence (33) the fact that convex combinations of qubit unital channels in EB^2 are not necessarily elements of the same set, can be traced back to the fact that the set formed by operators T satisfying the condition

$$\|T^2\|_1 \leq 1 \quad (61)$$

is not convex. To show this thesis it is sufficient to provide a counter-example. For instance take T as in (59) and its transpose T^\top . Since both these matrices satisfy the condition (34) they properly define two unital qubit channels Φ and Ψ via the correspondence (33). Furthermore since

$$\|T^2\|_1 = \|(T^\top)^2\|_1 = 0.98 < 1, \quad (62)$$

these maps belong to EB^2 . Take then the channel $\bar{\Phi} = (\Phi + \Psi)/2$: it is clearly unital and it is described by a matrix (33) formed by the convex convolution

$$\bar{T} = (T + T^\top)/2 = \begin{bmatrix} 0 & 0.615 & 0 \\ 0.615 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}. \quad (63)$$

Computing its $\|\bar{T}^2\|_1$ we notice however that it does not fulfill (62), indeed

$$\|\bar{T}^2\|_1 \simeq 1.01 > 1, \quad (64)$$

implying that $\bar{\Phi}$ is not in EB^2 , completing hence the proof.

b. Non convexity of EB^3 : A counter-example can be obtained by taking the unital channels Φ and Ψ , and $\bar{\Phi} = (\Phi + \Psi)/2$ associated with matrices T , T^\top , and $\bar{T} = (T + T^\top)/2$ where now however T is expressed as $O\Lambda$ with O is again as in Eq. (58) but with Λ being

$$\Lambda = \begin{bmatrix} 0.91 & 0 & 0 \\ 0 & 0.6 & 0 \\ 0 & 0 & 0.55 \end{bmatrix}. \quad (65)$$

With this choice in fact we have $\|T^3\|_1 = \|(T^\top)^3\|_1 \simeq 0.991 < 1$ (hence Φ and Ψ are in EB^3) but $\|\bar{T}^3\|_1 \simeq 1.03 > 1$ which instead implies that $\bar{\Phi} \notin \text{EB}^3$.

The example (63) seems to contradict one of the claims of [15] which states that the set of unital qubit EB^2 is closed under convex convolution. As for the incompatibility with [9] underlined at the end of Sec. III A 2, this originates from the fact that Ref. [15] always assumes the canonical structure (42) when analyzing the unital qubit channels. Indeed, even though, in general the set of unital qubit channels which are EB^n is not convex, it is true that the subset formed by qubit unital maps of EB^n which are in polar (or canonical) forms is convex. For $n = 2$ this was explicitly proved in [15]. Here we generalize this fact to arbitrary n . Let then Φ_{p_1} and Φ_{p_2} be unital qubit maps in polar form which are EB^n , that is $\|\Lambda_1^n\|_1 \leq 1$ and $\|\Lambda_2^n\|_1 \leq 1$ (Λ_1 and Λ_2 being the non-negative matrices that define the two maps within the correspondence (33)). According to the Hölder inequality [22] we have that given an arbitrary product $\Lambda_{i_1}\Lambda_{i_2}\dots\Lambda_{i_n}$ with $\{i_\alpha\}_{1 \leq \alpha \leq n} = 1, 2$, one has

$$\begin{aligned} \|\Lambda_{i_1}\Lambda_{i_2}\dots\Lambda_{i_n}\|_1 &\leq \|\Lambda_{i_1}\|_n \|\Lambda_{i_2}\|_n \dots \|\Lambda_{i_n}\|_n \\ &= \text{Tr}[\Lambda_{i_1}^n]^{\frac{1}{n}} \text{Tr}[\Lambda_{i_2}^n]^{\frac{1}{n}} \dots \text{Tr}[\Lambda_{i_n}^n]^{\frac{1}{n}} \\ &= \left(\|\Lambda_{i_1}^n\|_1 \|\Lambda_{i_2}^n\|_1 \dots \|\Lambda_{i_n}^n\|_1 \right)^{1/n} \\ &\leq 1. \end{aligned} \quad (66)$$

Thus, exploiting the sub-additivity of the trace norm, we get that for arbitrary probabilities q_1, q_2 we have

$$\begin{aligned} \|(q_1\Lambda_1 + q_2\Lambda_2)^n\|_1 &\leq \sum_{i_1=1}^2 \dots \sum_{i_n=1}^2 q_{i_1} \dots q_{i_n} \|\Lambda_{i_1} \dots \Lambda_{i_n}\|_1 \\ &\leq (q_1 + q_2)^n = 1, \end{aligned} \quad (67)$$

which proves that the mixed channel $q_1\Phi_{p_1} + q_2\Phi_{p_2}$ is an element of EB^n . The convexity of the set of unital qubit maps in canonical form can be proved in an similar way.

4. Geometry of unital EB^n maps in canonical form

For unital maps in canonical form (42), the necessary and sufficient condition (34) for the complete positivity

can be expressed as [18]

$$|\lambda_1 \pm \lambda_2| \leq |1 \pm \lambda_3|, \quad (68)$$

which, within the parameter space of the real 3-D vectors $(\lambda_1, \lambda_2, \lambda_3)$, identifies a tetrahedron characterized by the extreme points $(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)$. Similarly from Eqs. (37) and (41) we have that the set of unital entanglement-breaking maps EB for qubits in canonical form is restricted to the octahedron of vertices $(\pm 1, 0, 0)$ and permutations, which correspond to the intersections between the tetrahedron and its inversion through the origin [18]. The generalization to the case EB^n is straightforward. Indeed from Eqs. (40) and (53) it immediately follows that the canonical forms Φ_λ which are EB^n are identified by the intersection between the tetrahedron and the region

$$\sum_{0 \leq i \leq 3} |\lambda_i|^n \leq 1, \quad (69)$$

(for instance, the set EB^2 corresponds to the intersection between the tetrahedron and the unit ball in the origin, see also Fig. 2 of Ref. [15]). For sake of simplicity, in Fig. 1 we have reduced the tridimensional $(\lambda_1, \lambda_2, \lambda_3)$'s space to the space (λ_1, λ_2) by fixing $\lambda_3 = 1/2$, and studied the domain associated to the sets EB^n , $n \geq 1$. The shaded region (orange in online version), refers to the unital maps – i.e. it is a section of the tetrahedron (68). Notice that while the EB square is contained in the CPT set, this is not the case for higher order entanglement-breaking maps. The corners of the CPT rectangle, $\lambda_1 = \pm \lambda_3$ with $\lambda_2 = \pm 1$ and $\lambda_2 = \pm \lambda_3$ with $\lambda_1 = \pm 1$, refer to unital maps which can never belong to EB^n for finite λ_3 , as the EB^n condition reduces to $2|\lambda_3|^n \leq 0$, $n \geq 1$. Finally, observe that $\text{EB}^n \subset \text{EB}^{n+1}$, $n \geq 1$ (see Eq. (24)).

B. Generalized amplitude-damping channels

A generalized qubit amplitude-damping channel $\Psi_{p,\gamma}$ describes the physical process through which the system approaches equilibrium with its environment by spontaneous emission at finite temperature [4]. This is done by modifying the statistics of the populations associated to the ground state $|0\rangle$ and the excited state $|1\rangle$ which, in the following we identify with the eigenvectors of the Pauli matrix σ_z . We can represent them as

$$\Psi_{p,\gamma}[\rho] = \sum_{1 \leq i \leq 4} E_i \rho E_i^\dagger, \quad (70)$$

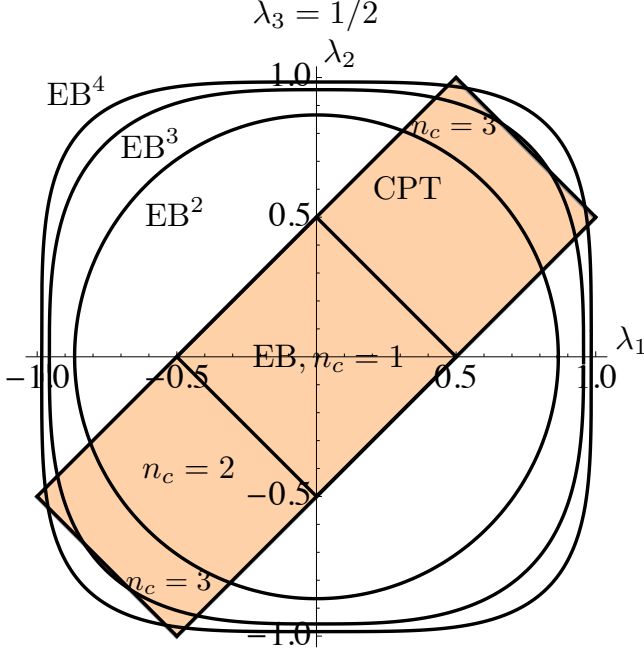


FIG. 1: (Color online) Section of the regions (69) which, for qubit maps in canonical form (42), identify the entanglement-breaking qubit maps of order n (here we set $\lambda_3 = 1/2$). The shaded region (orange in the online version) refers to the tetrahedron (68) which identifies all CPT maps Φ_λ . The domain associated to the EB set is the square at the center of the figure and corresponds to the section of the octahedron [18]. It is the only region (69) which is completely contained in the tetrahedron. The circle of radius $\sqrt{3}/2$ identifies instead the condition for Φ_λ to be EB^2 .

where E_i are the Kraus operators [24] which, in the canonical basis $\{|0\rangle, |1\rangle\}$, are expressed by the matrices

$$\begin{aligned} E_1 &= \sqrt{\gamma} \begin{pmatrix} 1 & 0 \\ 0 & \sqrt{1-p} \end{pmatrix}, \quad E_2 = \sqrt{\gamma} \begin{pmatrix} 0 & \sqrt{p} \\ 0 & 0 \end{pmatrix} \\ E_3 &= \sqrt{1-\gamma} \begin{pmatrix} \sqrt{1-p} & 0 \\ 0 & 1 \end{pmatrix}, \quad E_4 = \sqrt{1-\gamma} \begin{pmatrix} 0 & 0 \\ \sqrt{p} & 0 \end{pmatrix}, \end{aligned} \quad (71)$$

with $0 \leq p \leq 1$ and $0 \leq \gamma \leq 1$.

For these channels we will provide an explicit expression for the functional $\mu_c(\Psi_{p,\gamma})$ and compare it with $\mu_c(\Psi_{p,\gamma}^2)$ verifying Eq. (21). We will also compute explicitly the value of $n_c(\Psi_{p,\gamma})$ and construct examples of amendable channels of arbitrary order. Before entering in the details of these calculations however we observe that from Eq. (71) it follows

$$\Psi_{p,1-\gamma} = \mathcal{S}_1 \circ \Psi_{p,\gamma} \circ \mathcal{S}_1^\dagger, \quad (72)$$

where \mathcal{S}_1 is the Pauli rotation (48). This implies that we can restrict the analysis to the case $\gamma \in [0, 1/2]$ since the

following identities hold

$$\mu_c(\Psi_{p,1-\gamma}^n) = \mu_c(\Psi_{p,\gamma}^n), \quad \forall n, \quad (73)$$

$$n_c(\Psi_{p,1-\gamma}) = n_c(\Psi_{p,\gamma}), \quad (74)$$

(the first deriving from Eq. (10) while the second from (26)).

1. Noise addition via convex convolution

By definition, in order to determine μ_c we have to perform two minimizations: the first leads to $\mu(\Psi_{p,\gamma}; \rho_0)$, and the second consists of a minimization over all possible choices of ρ_0 (see Eqs. (4) and (6)). Regarding the second step it is useful to observe that for the maps (70) we can focus on ρ_0 lying on the z axis, i.e. on states with the following Bloch form

$$\rho_0^z = \frac{\mathbb{1} + v_z \sigma_z}{2}. \quad (75)$$

This can be proved following the same line of reasoning that led us to the identity (46) via Eq. (47). Indeed we notice that the channel $\Psi_{p,\gamma}$ commutes with all the unitary rotations $U_z = \exp[i\theta\sigma_z]$ around the z axis, i.e.

$$\Psi_{p,\gamma} = \mathcal{U}_z^\dagger \circ \Psi_{p,\gamma} \circ \mathcal{U}_z, \quad (76)$$

being $\mathcal{U}_z[\rho] := U_z \rho U_z^\dagger$. We can thus conclude that if there is a state $\rho_0 = (\mathbb{1} + \vec{v} \cdot \vec{\sigma})/2$ with arbitrary $\vec{v} \in \mathbb{R}^3$ such that $\Gamma_{\rho_0,\mu}^{\Psi_{p,\gamma}}$ is separable for some μ , then also the state $\int dm(U_z)(\mathcal{U}_z \otimes \mathcal{U}_z)[\Gamma_{\rho_0,\mu}^{\Psi_{p,\gamma}}]$ obtained by mixing over all the local rotations $\mathcal{U}_z \otimes \mathcal{U}_z$ will be separable ($dm(U_z)$ being the measure over the set of unitary rotations U_z). The latter however is given by

$$\begin{aligned} \int dm(U_z)(\mathcal{U}_z \otimes \mathcal{U}_z)[\Gamma_{\rho_0,\mu}^{\Psi_{p,\gamma}}] &= (1-\mu)(\Psi_{p,\gamma} \otimes I)[\psi_+] \\ &+ \mu \left(\int dm(U_z) \mathcal{U}_z[\rho_0] \otimes \frac{\mathbb{1}}{2} \right), \end{aligned} \quad (77)$$

which coincides with $\Gamma_{\rho_0^z,\mu}^{\Psi_{p,\gamma}}$ with ρ_0^z being the density matrix of the form (75) defined by

$$\int dm(U_z) \mathcal{U}_z[\rho_0] = \rho_0^z. \quad (78)$$

Thus if $\Gamma_{\rho_0,\mu}^{\Psi_{p,\gamma}} \in \text{SEP}$, then $\Gamma_{\rho_0^z,\mu}^{\Psi_{p,\gamma}} \in \text{SEP}$ which implies $\mu(\Phi; \rho_0^z) \leq \mu(\Phi; \rho_0)$ as anticipated.

To determine $\mu(\Psi_{p,\gamma}; \rho_0^z)$, we recall that for two qubits the state $\Gamma_{\rho_0^z,\mu}^{\Psi_{p,\gamma}}$ is separable iff, when partially transposed, it shows a positive determinant [25]. Exploiting this we can then write

$$\mu(\Psi_{p,\gamma}; \rho_0^z) = \frac{p[4p(\gamma-1)\gamma - 2\gamma v_z + v_z - 3] + 4 - \sqrt{p^2(v_z - 2\gamma + 1)^2 + 4p(v_z^2 - 1) - 4v_z^2 + 4}}{4p^2(\gamma-1)\gamma + 2p(-2v_z\gamma + v_z - 1) + v_z^2 + 3}, \quad (79)$$

which needs then to be minimized over $v_z \in [-1, 1]$. Remembering that $\gamma \leq 1/2$, we find that for

$$p \leq \bar{p} := \frac{\sqrt{4\gamma^2 - 8\gamma + 5} - 1}{2(1-\gamma)^2}, \quad (80)$$

the function (79) has a global minimum at $v_z = \bar{v}_z \in]-1, 1[$

$$\bar{v}_z(\gamma, p) = \frac{p(p + 2\sqrt{1-p})(1-2\gamma)}{4 - p(p+4)}, \quad (81)$$

yielding

$$\mu_c(\Psi_{p \leq \bar{p}, \gamma \leq 1/2}) = \frac{p^2 + 3p + 2\sqrt{1-p} - 4}{p^2 + 2p - 3}, \quad (82)$$

(notice that in this case, $\mu_c(\Psi_{p,\gamma})$ does not depend on γ). Vice-versa, for $p \geq \bar{p}$ the quantity $\mu(\Psi_{p,\gamma}; \rho_0^z)$ is a monotonous decreasing function of v_z , whose minimum corresponds to $v_z = 1$ and is given by

$$\mu_c(\Psi_{p \geq \bar{p}, \gamma \leq 1/2}) = \frac{p[p(\gamma-1)\gamma - 1] + 1}{p\gamma[p(\gamma-1) - 1] + 1}. \quad (83)$$

The function $\mu_c(\Psi_{p,\gamma})$ is continuous at $p = \bar{p}$ where both (82) and (83) yield

$$\mu_c(\Psi_{\bar{p}, \gamma \leq 1/2}) = \frac{2 - 4\gamma}{-4\gamma + \sqrt{4(\gamma-2)\gamma + 5} + 3}, \quad (84)$$

which vanishes for $\gamma = 1/2$. In this case we have $\bar{p}(\gamma = 1/2) = 2(\sqrt{2} - 1) \simeq 0.828$ and $\Psi_{p,\gamma} \in \text{EB}$ (this point belongs to the curve $p = p_1(\gamma)$ computed in the next subsection, see Eq. (89)).

Similar results are found for $\Psi_{p,\gamma}^2 := \Psi_{p,\gamma} \circ \Psi_{p,\gamma}$. More precisely we have

$$\mu_c(\Psi_{p \leq \bar{p}, \gamma \leq 1/2}^2) = \frac{p^2 - 4p + 2}{p^2 - 4p + 3} \quad (85)$$

$$\mu_c(\Psi_{p \geq \bar{p}, \gamma \leq 1/2}^2) = \frac{(p-2)p[(p-2)p(\gamma-1)\gamma + 1] + 1}{(p-2)p\gamma[(p-2)p(\gamma-1) + 1] + 1}, \quad (86)$$

with

$$\bar{p}(\gamma) := \frac{\sqrt{4\gamma^2 - 8\gamma + 5} + 2\gamma - 3}{2(\gamma - 1)}. \quad (87)$$

Furthermore, $\mu_c(\Psi_{p,\gamma}^2)$ vanishes for $\{\bar{p}, \gamma\} = \{2 - \sqrt{2}, 1/2\} \simeq \{0.586, 1/2\}$. We also find that incidentally the following identity holds, $\mu_c(\Psi_{\bar{p},\gamma}) = \mu_c(\Psi_{\bar{p},\gamma}^2)$. The analysis gives analogous results for $\Psi_{p,\gamma}^n$, $n > 3$. In the inset in Fig. 2, as an example, we have shown the behavior of $\mu_c(\Psi_{p,\gamma})$ and $\mu_c(\Psi_{p,\gamma}^2)$ for $\gamma = 1/3$.

2. Amendable channels of arbitrary order

By requiring the positivity of the determinant of the partially transposed Choi-Jamiolkowski state associated to this map [25], we can determine the region of the parameter space $\{p, \gamma\}$ such that $\Psi_{p,\gamma}^n \in \text{EB}$, see e.g. Fig. 2. A direct calculation yields that $n_c(\Psi_{p,\gamma}) = n \geq 1$ if and only if

$$p_n(\gamma) \leq p \leq p_{n-1}(\gamma), \quad (88)$$

where $p_0 = 1$ and

$$p_n(\gamma) = 1 - \left(1 - \frac{2}{1 + \sqrt{1 + 4\gamma(1-\gamma)}}\right)^{\frac{1}{n}}, \quad (89)$$

(see also [15] for $n = 1, 2$).

Starting from this observation we can now construct examples of amendable channels, by looking for maps of the form

$$\Psi_{p,\gamma}^{\mathcal{U}} = \Psi_{p,\gamma} \circ \mathcal{U}, \quad (90)$$

with \mathcal{U} being a unitary, such that $\Psi_{p,\gamma}^{\mathcal{U}}$ is entanglement-breaking of order 2 with $n_c(\Psi_{p,\gamma}^{\mathcal{U}}) = 2$, while $\Psi_{p,\gamma}$ is not. Accordingly, some simple algebra allows us to write

$$2 = n_c(\Psi_{p,\gamma}^{\mathcal{U}}) < n_c(\Psi_{p,\gamma}) = n_c(\mathcal{U}^\dagger \circ \Psi_{p,\gamma}^{\mathcal{U}}), \quad (91)$$

which provides an instance of Eq. (28) with the filtering transformation \mathcal{V} being now \mathcal{U}^\dagger (indeed for such examples we will have $\Psi_{p,\gamma}^{\mathcal{U}} \circ \mathcal{U}^\dagger \circ \Psi_{p,\gamma}^{\mathcal{U}} \notin \text{EB}$ proving that we have amended the action of the mapping $(\Psi_{p,\gamma}^{\mathcal{U}})^2$). The region of the parameter space $\{p, \gamma\}$ for which the inequality (91) holds for some \mathcal{U} is shown in gray (red in online version) in Fig. 3. A numerical investigation shows that it is given by the union of two areas. The first one contains all points for which (28) holds with \mathcal{U} being the Pauli rotation (48) \mathcal{S}_1 : this region can be determined analytically and it is delimited from below by the condition

$$p \geq \frac{1}{4(1-\gamma)\gamma} \left[-\sqrt{4(1-\gamma)\gamma + 1} + \sqrt{\left(1 - 2\sqrt{1 - 4(\gamma-1)\gamma}\right)(1-2\gamma)^2 + 1 + 1} \right], \quad (92)$$

(dashed curve in the figure). The second one instead contains all points for which (28) holds for $\mathcal{U} = \mathcal{R}_2(\pi/2) \circ \mathcal{R}_1(\pi/2)$ with \mathcal{R}_j is the super-operator associated with the rotations $\exp[-i(\pi/4)\sigma_j]$ (this has been characterized

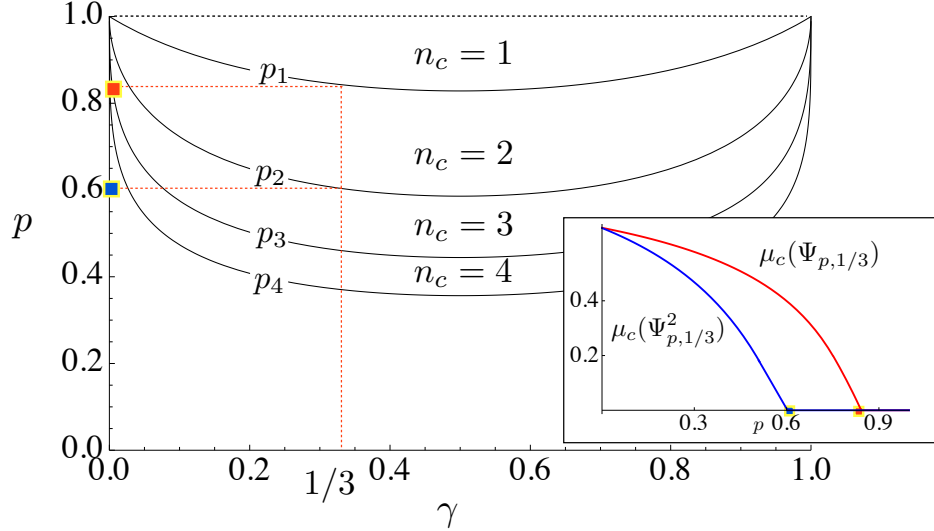


FIG. 2: Regions of the space $\{p, \gamma\}$ associated with channels $\Psi_{p,\gamma}$ which have $n_c(\Psi_{p,\gamma}) = n$ for some integer n (see Eqs. (88) and (89)). Notice that for all γ 's we have $p_n(\gamma) = p_n(1 - \gamma)$ in agreement with Eq. (74). Furthermore, amplitude-damping channels, corresponding to $\gamma = 0, 1$ do not become EB after any number of iterations. In the inset figure, is shown the dependence of $\mu_c(\Psi_{p,\gamma})$ and $\mu_c(\Psi_{p,\gamma}^2)$ on p , for $\gamma = 1/3$. This functional vanishes as soon as the map becomes entanglement-breaking. As expected, for all p 's we find $\mu_c(\Psi_{p,\gamma}^2) \leq \mu_c(\Psi_{p,\gamma})$ (see Eq. (21)).

only numerically and it is delimited from below by the dotted line of the figure).

Notice that for any given m integer the shaded area of Fig. 3 has a not-null overlap with the region (88) associated with channels $\Psi_{p,\gamma}$ having $n_c(\Psi_{p,\gamma}) = m$. A formal proof of this fact can be obtained by noticing that the boundary determined by Eq. (92) intercepts the vertical axis for $\{\gamma, p\} = \{0, (\sqrt{5} - 1)/2\}$ which, according to Eq. (89), does not belong the m -th region of Eq. (88). For these points the rhs of Eq. (91) is explicitly equal to m , providing hence direct evidence of the existence of amenable channels of order m (see Sec. II B 1).

In Fig. 4 we finally compare the values of $\mu_c(\Psi_{p,\gamma}^2)$ with that of $\mu_c(\Psi_{p,\gamma} \circ \mathcal{U} \circ \Psi_{p,\gamma})$ as functions of p having fixed $\mathcal{U} = \mathcal{S}_1$ and $\gamma = 1/10$ (part (a)), and $\mathcal{U} = \mathcal{R}_2(\pi/2) \circ \mathcal{R}_1(\pi/2)$ and $\gamma = 2/5$ (part (b)). In both cases the amended channels $\Psi_{p,\gamma}$ present an higher value of the functional μ_c .

IV. GAUSSIAN CHANNELS

In what follows we will show that behaviors similar to those seen in the previous section for finite dimensional systems, still hold also for continuous variables. In particular we will discuss how n_c can be used to classify the set of Gaussian maps [26] in a way which mimics what obtained for qubit channels.

Gaussian channels provide a mathematical description for the most common source of noise encountered in optical implementations, among which we have attenuation, amplification and thermalization processes [26]. They

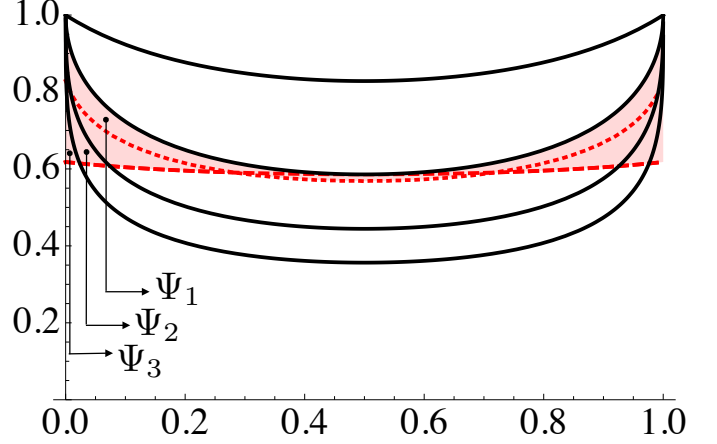


FIG. 3: (Color online) Amenable channels: the shaded region (red in online version) contains all points where (91) holds for some \mathcal{U} . The points Ψ_1 , Ψ_2 and Ψ_3 are examples of CPT maps which are amenable of order $m = 3$, $m = 4$ and $m = 5$ respectively (see text).

are CPT maps such that when operating on a Gaussian input state preserve its structure. In particular, a state $\rho \in \mathfrak{S}(\mathcal{H}_S)$ of a bosonic system with f degrees of freedom, is Gaussian if its characteristic function $\phi_\rho(z) = \text{Tr}[\rho W(z)]$ has a Gaussian form, where $W(z)$ is the unitary Weyl operator defined on the real vector space \mathbb{R}^{2f} , $W(z) := \exp[i\vec{R} \cdot \vec{z}]$, with $\vec{R} = \{Q_1, P_1, \dots, Q_f, P_f\}$ and Q_i, P_i are the canonical observables for the bosonic sys-

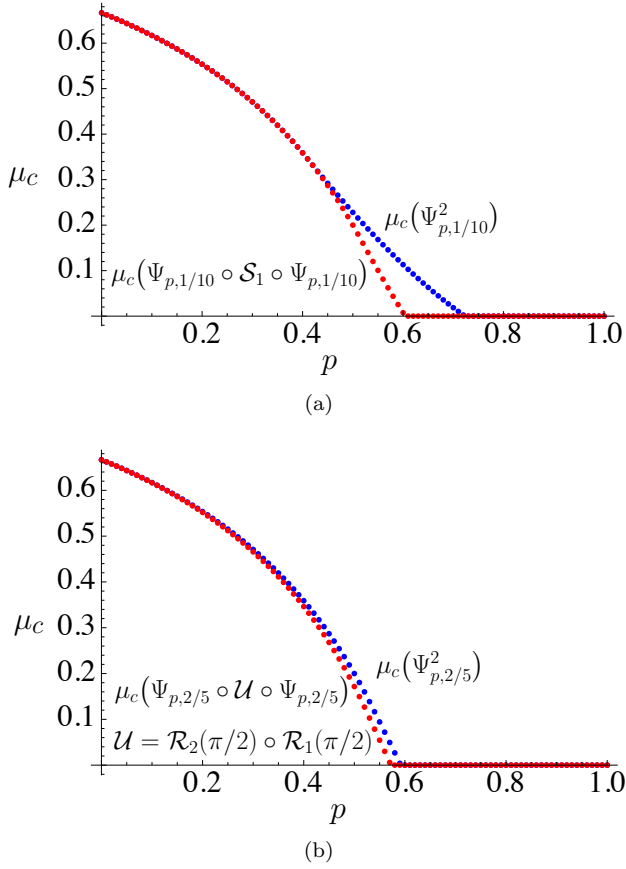


FIG. 4: (Color online) Plot of $\mu_c(\Psi_{p,\gamma}^2)$ and $\mu_c(\Psi_{p,\gamma} \circ \mathcal{U} \circ \Psi_{p,\gamma})$, with $\mathcal{U} = \mathcal{S}_1$ for $\gamma = 1/10$ (a) and $\mathcal{U} = \mathcal{R}_2(\pi/2) \circ \mathcal{R}_1(\pi/2)$ for $\gamma = 2/5$ (b). For this particular choice of unitary maps we have that $\mu_c(\Psi_{p,\gamma} \circ \mathcal{U} \circ \Psi_{p,\gamma}) \leq \mu_c(\Psi_{p,\gamma}^2)$.

tem. Gaussian maps Φ can be conveniently described in terms of the action of their duals Φ^* on the Weyl operator:

$$\Phi^*[W(z)] = W(Kz) \exp(il^\top z - \frac{1}{2}z^\top \beta z) \quad (93)$$

where $l \in \mathbb{R}^{2f}$, and K and β are $2f \times 2f$ matrices such that

$$\beta \geq \pm i[\Delta - K^\top \Delta K]/2, \quad (94)$$

Δ being the canonical symplectic form on \mathbb{R}^{2f} . The above inequality is the necessary and sufficient condition which guarantees the complete positivity of Φ . A Gaussian channel is therefore characterized by the triplet (K, l, β) . The composition of two Gaussian maps, $\Phi = \Phi_2 \circ \Phi_1$, described by (K_1, l_1, β_1) and (K_2, l_2, β_2) respectively, is still a Gaussian map whose parameters are given by

$$\Phi_2 \circ \Phi_1 \longrightarrow \begin{cases} K = K_1 K_2 \\ l = K_2^\top l_1 + l_2 \\ \beta = K_2^\top \beta_1 K_2 + \beta_2. \end{cases} \quad (95)$$

Finally, a Gaussian map Φ is entanglement-breaking [27]

if and only if

$$\beta = \alpha + \nu, \quad (96)$$

where

$$\alpha \geq \frac{i}{2}\Delta, \quad \text{and} \quad \nu \geq \frac{i}{2}K^\top \Delta K. \quad (97)$$

A. One-mode channels

For the case of continuous variables systems, one-mode Gaussian mappings ($f = 1$) can be seen as the analogous of the qubit channels for finite dimensional systems. In this section we will quantify the noise level of two representative examples of this class of maps, i. e. the attenuation and the amplification channels. By exploiting the conservation of the Gaussian character of these transformations under concatenation, and the entanglement-breaking conditions (96)-(97), in what follows we will focus on the classification of attenuation and amplification channels according to n_c , postponing the analysis of the addition via mixing to future works.

1. Attenuation channels

One-mode attenuation channels Φ_{At} are Gaussian mappings such that:

$$\Phi_{At} \longrightarrow \begin{cases} K = k\mathbb{1} \\ l = 0 \\ \beta = \left(N_0 + \frac{1-k^2}{2}\right)\mathbb{1} \end{cases} \quad (98)$$

where $\mathbb{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $k \in (0, 1)$ and $N_0 \geq 0$, the latter condition deriving from the CPT requirement (94) on Φ_{At} . From Eq. (96) it follows that $\Phi_{At} \in \text{EB}$ and $n_c(\Phi_{At}) = 1$ iff

$$\beta \geq \frac{i}{2}(1+k^2)\Delta \iff N_0 \geq k^2. \quad (99)$$

In order to generalize the condition above to $n_c(\Phi_{At}) = n$ with $n \geq 2$, let us start by studying the composite map $\Phi_{At}^2 := \Phi_{At} \circ \Phi_{At}$. It is given by

$$\Phi_{At}^2 \longrightarrow \begin{cases} K_2 = k^2\mathbb{1} \\ l_2 = 0 \\ \beta_2 = (1+k^2)\left(N_0 + \frac{1-k^2}{2}\right)\mathbb{1}. \end{cases} \quad (100)$$

Notice that if we define

$$k_2 := k^2 \leq 1 \quad \text{and} \quad (N_0)_2 := N_0(1+k^2) \geq 0 \quad (101)$$

we get

$$\beta_2 = \left((N_0)_2 + \frac{1-k_2^2}{2}\right)\mathbb{1}, \quad (102)$$

which plays the same role of β for Φ_{At} . It immediately follows that $\Phi_{At}^2 \in \text{EB}$ if and only if

$$\beta_2 \geq \frac{i}{2}(k_2^2 + 1)\Delta \iff (N_0)_2 \geq k_2^2 \quad (103)$$

(see Eq. (99)), that is $n_c(\Phi_{At}) = 2$ for

$$\frac{k^4}{1+k^2} \leq N_0 \leq k^2. \quad (104)$$

We can now generalize these results to the case $n_c(\Phi_{At}) = n \geq 2$. From Eq. (95) we get

$$\Phi_{At}^n \longrightarrow \begin{cases} K_n = k_n \mathbb{1} \\ l_n = 0 \\ \beta_n = \left((N_0)_n + \frac{1-k_n^2}{2} \right) \mathbb{1}, \end{cases} \quad (105)$$

where

$$k_n := k^n \leq 1 \quad \text{and} \quad (N_0)_n := N_0 \sum_{j=0}^{n-1} k^{2j} \geq 0. \quad (106)$$

It follows that $\Phi_{At}^n \in \text{EB}$ iff $(N_0)_n \geq k_n^2$, and $n_c(\Phi_{At}) = n$ if and only if

$$\frac{k^{2n}}{\sum_{j=0}^{n-1} k^{2j}} \leq N_0 \leq \frac{k^{2(n-1)}}{\sum_{j=0}^{n-2} k^{2j}}. \quad (107)$$

2. Amplification channels

One-mode amplification maps are very similar to the attenuation channels, where now $k \geq 1$ and

$$\Phi_{Am} \longrightarrow \begin{cases} K = k \mathbb{1} \\ l = 0 \\ \beta = \left(N_0 + \frac{k^2-1}{2} \right) \mathbb{1}. \end{cases} \quad (108)$$

From Eq. (96) we have that the entanglement-breaking condition for this class of maps is

$$\beta \geq \frac{i}{2}(1+k^2)\Delta \iff N_0 \geq 1. \quad (109)$$

For $n \geq 2$ one has

$$\Phi_{Am}^n \longrightarrow \begin{cases} K_n = k_n \mathbb{1} \\ l_n = 0 \\ \beta_n = \left((N_0)_n + \frac{k_n^2-1}{2} \right) \mathbb{1}, \end{cases} \quad (110)$$

with k_n and $(N_0)_n$ given by Eq. (106). It follows that the composite map Φ_{Am}^n is entanglement-breaking iff $(N_0)_n \geq 1$, and $n_c(\Phi_{Am}) = n \geq 2$ for

$$\left(\sum_{j=0}^{n-1} k^{2j} \right)^{-1} \leq N_0 \leq \left(\sum_{j=0}^{n-2} k^{2j} \right)^{-1}. \quad (111)$$

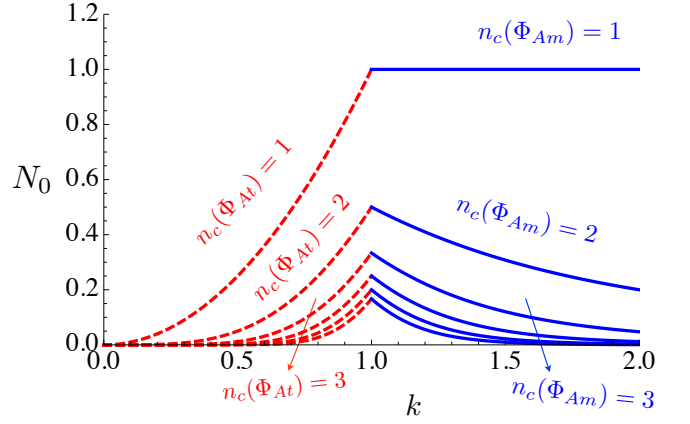


FIG. 5: (Color online) Boundaries of the regions in the parameter space $\{k, N_0\}$ such that $n_c(\Phi_{At}) = n$ (dashed red lines), and $n_c(\Phi_{Am}) = n$ (solid blue lines) for $n = 1, \dots, 6$.

Analogously to what done in Fig. 2 for the case of generalized amplitude-damping channels, in Fig. 5 we propose a scan of the parameter space $\{k, N_0\}$ for attenuation and amplification maps according the n_c criterion by plotting the boundaries of the regions such that $n_c(\Phi_{At}) = n$ (dashed red lines), and $n_c(\Phi_{Am}) = n$ (solid blue lines) for $n = 1, \dots, 6$. Notice that, the region such that $\Phi_{At/Am} \in \text{EB}^n$ grows with n , as it is limited from below by the curves

$$\begin{cases} N_0 = k^2, & n = 1 \\ N_0 = \frac{k^{2n}}{\sum_{j=0}^{n-1} k^{2j}}, & n \geq 2 \end{cases} \quad (112)$$

and

$$\begin{cases} N_0 = 1, & n = 1 \\ N_0 = \left(\sum_{j=0}^{n-1} k^{2j} \right)^{-1}, & n \geq 2 \end{cases} \quad (113)$$

for attenuation and amplification channels, respectively. See Eqns. (99), (107), (109) and (111).

V. CONCLUSIONS

In our analysis we have introduced two new functionals (μ_c and n_c) that can be used to quantify the noise level of a map. These have been characterized in terms of general properties and have been explicitly evaluated for some class of channels. Along the way we have also introduced the notion of amendable channels showing how there exist maps which can be prevented from becoming entanglement-breaking after subsequent applications by interposing some extra transformations. For the sake of simplicity in our study we have only considered the case in which the *same* unitary transformation is interposed between successive channel uses. This however is not the only possibility and it is an interesting question of quantum control to determine what are the optimal

operations one has to perform in order to guarantee that after certain number of reiterations entanglement will not be destroyed in the system.

We finally notice that when introducing the functionals n_c and μ_c we started from the identification of the entanglement-breaking channels as a benchmark set to evaluate the noise level of other quantum transformations (the choice being operationally motivated by the extreme deteriorating effects that EB maps have on the system). Again this however is not the only possibility. For instance another reasonable choice is to replace the entanglement-breaking channels with the set formed by the Positive Partial Transpose (or *binding*) maps [28] producing new noise measures $\mu_c^{(PPT)}$ and $n_c^{(PPT)}$ for Φ . We remind that a channel is said to be PPT if when extended on an ancilla A the only entanglement one can find in the output of $S + A$ is non-distillable [13]. Similarly to the case of EB maps, also the PPT set is stable under convex convolution and iteration: therefore $\mu_c^{(PPT)}$ and $n_c^{(PPT)}$ will again assume the minimum allowed values on the benchmark set guarantying that they are well defined quantities for all Φ . Now, since for qubits the entanglement is always distillable [29], when S has dimension 2 we have $\mu_c^{(PPT)}(\Phi) = \mu_c(\Phi)$ and $n_c^{(PPT)}(\Phi) = n_c(\Phi)$ (a property we explicitly exploited when computing the values of μ_c and n_c for qubit channels). On the contrary when S has larger dimensions, $\mu_c^{(PPT)}(\Phi)$ and $n_c^{(PPT)}(\Phi)$ need not to reduce to $\mu_c(\Phi)$ and $n_c(\Phi)$ yielding a qualitatively new way of gauging the noise level of Φ .

Acknowledgments

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Appendix A: A convexity property of $\mu(\Phi; \rho_0)$

In this appendix we show a convexity rule for the functional $\mu(\Phi; \rho_0)$ with respect to its second argument.

Consider a generic statistical ensemble $\{p_j; \Phi_{\text{DEP}}^{\rho_j}\}$ of completely depolarizing maps $\Phi_{\text{DEP}}^{\rho_i}$ distributed according to the probabilities p_j . We will prove that $\mu(\Phi; \rho_0)$ is convex with respect to its second argument:

$$\mu(\Phi; \sum_i p_i \rho_i) \leq \tilde{\mu} \leq \sum_i p_i \mu_{\text{DEP}}^j, \quad (\text{A1})$$

where $\mu_{\text{DEP}}^j := \mu(\Phi; \rho_j)$, $\rho_j \in \mathfrak{S}(\mathcal{H}_S)$ and

$$\tilde{\mu} = \left(\sum_i \frac{p_i}{\mu_{\text{DEP}}^j} \right)^{-1}. \quad (\text{A2})$$

In order to prove the first inequality we define the probabilities $\tilde{q}_i := \frac{p_i}{\mu_{\text{DEP}}^j} / (\sum_j \frac{p_j}{\mu_{\text{DEP}}^j})$ and notice that

$$\sum_j \tilde{q}_j \Gamma_{\rho_j, \mu_{\text{DEP}}^j}^{\Phi} = (1 - \tilde{\mu})(\Phi \otimes I)[\psi_+] + \tilde{\mu} \left(\sum_i p_i \rho_i \otimes \mathbb{1}/d \right)$$

is separable. The second inequality in (A1) is equivalent to

$$\prod_{\ell} \mu_{\text{DEP}}^{\ell} \leq \left(\sum_i p_i \prod_{j \neq i} \mu_{\text{DEP}}^j \right) \left(\sum_k p_k \mu_{\text{DEP}}^k \right) \quad (\text{A3})$$

and can be easily proved applying the normalization condition $\sum p_i = 1$.

Appendix B: A decomposition for unital qubit maps

In this appendix we verify the commutation rule (47) by noticing that for unital qubit maps in canonical form, Φ_{λ} there exists a decomposition in terms of the super-operators (48), i.e.

$$\Phi_{\lambda} = \sum_{0 \leq i \leq 3} p_i \mathcal{S}_i, \quad (\text{B1})$$

with p_i being real quantities (as a matter of fact p_i can be shown to be probabilities [18]: here however we will not need this property). From this then (47) simply follows from the fact that for all i, j we have $\mathcal{S}_j \circ \mathcal{S}_i \circ \mathcal{S}_j = \mathcal{S}_i$.

To see Eq. (B1) we expand the input state ρ in the Bloch sphere formalism as $\rho = (1 + \vec{v} \cdot \vec{\sigma})/2$ and use the fact that

$$\Phi_{\lambda}[\rho] = \frac{1}{2} \sum_{0 \leq i \leq 3} \lambda_i v_i \sigma_i, \quad (\text{B2})$$

where we set $\lambda_0 = v_0 = 1$. On the other hand, if we apply the super-operator $\sum_{0 \leq i \leq 3} p_i \mathcal{S}_i$ to ρ we get

$$\sum_{0 \leq i \leq 3} p_i \mathcal{S}_i[\rho] = \frac{1}{2} \sum_{0 \leq i, j \leq 3} p_i M_{ij} v_j \sigma_j, \quad (\text{B3})$$

where M_{ij} are elements of the 4×4 invertible symmetric real matrix

$$M := \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}, \quad (\text{B4})$$

defined by the relation $M_{ij} \sigma_j = \mathcal{S}_i(\sigma_j) = \sigma_i \sigma_j \sigma_i$. By equating Eqs. (B2) and (B3) we conclude that Eq. (B1) holds if and only if

$$\vec{p} = M^{-1} \vec{\lambda}, \quad (\text{B5})$$

where here \vec{p} and $\vec{\lambda}$ stand for the 4-dimensional vectors (p_0, \dots, p_3) and $(\lambda_0, \dots, \lambda_3)$, respectively.

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